

A Simple Proof of Fast Polarization

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Abstract

Fast polarization is an important and useful property of polar codes. It was proved for the binary polarizing 2×2 kernel by Arikan and Telatar. The proof was later generalized by Şaşıoğlu. We give a simplified proof.

Index Terms

polar codes, fast polarization

I. INTRODUCTION

Polar codes are a novel family of error correcting codes, invented by Arikan [1]. The seminal definitions and assumptions in [1] were soon expanded and generalized. Key to almost all the results involving polar codes is the concept of fast polarization. The essence of fast polarization is the phenomenon stated in the following lemma. The lemma was used implicitly by Korada, Şaşıoğlu, and Urbanke [2, proof of Theorem 11], and is a generalization of a result by Arikan and Telatar [3, Theorem 3]. Its explicit formulation and full proof are due to Şaşıoğlu [4, Lemma 5.9].

Lemma 1: Let B_0, B_1, \dots be an i.i.d. process where B_0 is uniformly distributed over $\{1, 2, \dots, \ell\}$. Let Z_0, Z_1, \dots be a $[0, 1]$ -valued random process where

$$Z_{m+1} \leq K \cdot Z_m^{D_t}, \quad \text{whenever } B_m = t. \quad (1)$$

We assume $K \geq 1$ and $D_1, D_2, \dots, D_\ell > 0$. Suppose also that Z_m converges almost surely to a $\{0, 1\}$ -valued random variable Z_∞ . Then, for any

$$0 < \beta < E \triangleq \frac{1}{\ell} \sum_{t=1}^{\ell} \log_{\ell} D_t$$

we have

$$\lim_{m \rightarrow \infty} \Pr[Z_m \leq 2^{-\ell^{\beta \cdot m}}] = \Pr[Z_\infty = 0]. \quad (2)$$

The lemma is used to prove that the Bhattacharyya parameter associated with a random variable that underwent polarization (for example, a synthesized channel) polarizes to 0 at a rate faster than polynomial [4, Theorem 5.4]. A similar claim holds in the case of polarization of the Bhattacharyya parameter to 1 [5, Theorem 16].

The original proof [4, Lemma 5.9] of Lemma 1 is somewhat involved. To summarize, if K were equal to 1, the proof would follow almost directly from the strong law of large numbers. However, for $K > 1$, a sequence

of bootstrapping arguments is applied. That is, a current bound on the rate of convergence of Z_m to 0 is used to derive a stronger bound, and the process is repeated.

The main aim of this paper is to give a simpler proof of Lemma 1. Thus, we hopefully give insight into the simple mechanics that are at play. Our simpler proof also leads to a stronger result. That is, we will prove the following, which implies Lemma 1.

Lemma 2: Let $\{B_m\}_{m=0}^\infty$, $\{Z_m\}_{m=0}^\infty$, K , and E be as in Lemma 1. Then, for $0 < \beta < E$,

$$\lim_{m_0 \rightarrow \infty} \Pr[Z_m \leq 2^{-\ell^{\beta \cdot m}} \text{ for all } m \geq m_0] = \Pr[Z_\infty = 0]. \quad (3)$$

Note that Lemma 2 has an “almost sure flavor” [6, page 69, Equation (2)], while Lemma 1 has an “in probability flavor” [6, page 70, Equation (5)]. We prove Lemma 2 in Section II and show that it implies Lemma 1 in Section III.

II. PROOF OF LEMMA 2

Let $\epsilon_a, \epsilon_b > 0$ and $m_a < m_b$ be given constants, specified towards the end. We now define three events, denoted A , B , and C .

$$A : \quad |Z_m - Z_\infty| \leq \epsilon_a, \quad \text{for all } m \geq m_a. \quad (4)$$

$$B : \quad \left| \frac{|\{m_a \leq i < m : B_i = t\}|}{m - m_a} - \frac{1}{\ell} \right| \leq \epsilon_b, \quad \text{for all } m \geq m_b \text{ and } 1 \leq t \leq \ell. \quad (5)$$

$$C : \quad Z_\infty = 0. \quad (6)$$

Recall that the Z_m converge almost surely to Z_∞ . Thus, essentially by definition (see [6, Theorem 4.1.1.]), we have for any fixed $\epsilon_a > 0$ that

$$\lim_{m_a \rightarrow \infty} \Pr[A] = 1.$$

Note that event B is concerned with the frequency of t in the subsequence of i.i.d. random variables $B_{m_a}, B_{m_a+1}, \dots, B_{m-1}$, which are uniform over $\{1, 2, \dots, \ell\}$. Thus, by the strong law of large numbers¹ [6, Theorem 5.4.2.], we have for any fixed ϵ_b and m_a that

$$\lim_{m_b \rightarrow \infty} \Pr[B] = 1.$$

We deduce that for any $\delta_a, \delta_b > 0$ there exist $m_a < m_b$ such that

$$\Pr[A] \geq 1 - \delta_a \quad (7)$$

and

$$\Pr[B] \geq 1 - \delta_b. \quad (8)$$

Hence,

$$\Pr[A \cap B \cap C] \geq \Pr[Z_\infty = 0] - \delta_a - \delta_b. \quad (9)$$

¹The strong law of large numbers is applied ℓ times. Each application is with respect to the indicators $B_i = t$, where $1 \leq t \leq \ell$. As before, we use [6, Theorem 4.1.1.].

Let us see what the event $A \cap B \cap C$ implies. For fixed $0 < \epsilon_a, \epsilon_b, \delta_a, \delta_b < 1$, let m_a and m_b be as above. Define the shorthand

$$\theta \triangleq -\log_{\epsilon_a} K.$$

Note that θ is non-negative, and approaches 0 as ϵ_a approaches 0. By the definition of the events A and C , we have that $Z_m \leq \epsilon_a$ when $m \geq m_a$. Thus, $K \leq Z_m^{-\theta}$ when $m \geq m_a$. Hence, we simplify (1) to

$$Z_{m+1} \leq Z_m^{D_t - \theta}, \quad \text{whenever } m \geq m_a \text{ and } B_m = t. \quad (10)$$

The above equation is the heart of the proof: we have effectively managed to “make K equal 1” — the simple case discussed earlier. We have “paid” for this simplification by having the exponents be $D_t - \theta$ instead of the original D_t . However, since θ can be made arbitrarily close to 0, this will not be a problem. Essentially, all that remains is some simple algebra, followed by taking the relevant parameters small/large enough. We do this now.

Let us assume ϵ_a is small enough such that $D_t - \theta > 0$ for all $1 \leq t \leq \ell$ and that $\epsilon_b < 1/\ell$. Recall also that $Z_{m_a} \in [0, 1]$. Combining (10) with event B , we deduce that for all $m \geq m_b$,

$$Z_m \leq Z_{m_a}^{\prod_{t=1}^{\ell} (D_t - \theta)^{(m - m_a) \cdot (1/\ell \pm \epsilon_b)}}, \quad (11)$$

where the above “ \pm ” notation is in fact a function of t , defined as

$$\pm \triangleq \begin{cases} + & \text{if } D_t - \theta \leq 1, \\ - & \text{otherwise.} \end{cases}$$

By the definition of event A , we have that $Z_{m_a} \leq \epsilon_a$. We will further assume that $\epsilon_a \leq 1/2$. Hence, (11) simplifies to the claim that for all $m \geq m_b$,

$$Z_m \leq 2^{-\prod_{t=1}^{\ell} (D_t - \theta)^{(m - m_a) \cdot (1/\ell \pm \epsilon_b)}} = 2^{-\ell(E - \Delta)m}, \quad (12)$$

where

$$\Delta = \sum_{t=1}^{\ell} \frac{1}{\ell} \log_{\ell} \left(\frac{D_t}{D_t - \theta} \right) - \sum_{t=1}^{\ell} \pm \epsilon_b \log_{\ell} (D_t - \theta) + \sum_{t=1}^{\ell} \frac{m_a}{m} \left(\frac{1}{\ell} \pm \epsilon_b \right) \log_{\ell} (D_t - \theta). \quad (13)$$

In light of (3), our goal is to show that for a given $\beta < E$ and $\delta_a, \delta_b > 0$, we can choose $m_a < m_b$, and $\epsilon_a, \epsilon_b > 0$ as above such that $\Delta < E - \beta$. We do this by showing that each of the three sums in (13) can be made smaller than $(E - \beta)/3$. Recalling that θ goes to 0 as ϵ_a tends to 0, we deduce that the first sum can be made smaller than $(E - \beta)/3$ by taking ϵ_a small enough. Similarly, we can make the second sum smaller than $(E - \beta)/3$ by taking ϵ_b small enough. For the third sum, we first fix m_a large enough such that (7) holds (note that event A is a function of ϵ_a , which is by now fixed). Lastly, we take m_b large enough such that the third sum is smaller than $(E - \beta)/3$ for all $m \geq m_b$, and (8) holds (again, note that event B is a function of m_a and ϵ_b , which have been fixed).

Recall that our aim is to prove (3). We deduce from (9), (12), and the above paragraph that for all $\delta_a, \delta_b > 0$ and $0 < \beta < E$,

$$\lim_{m_0 \rightarrow \infty} \overbrace{\Pr[Z_m \leq 2^{-\ell^{\beta \cdot m}} \text{ for all } m \geq m_0]}^D \geq \Pr[Z_{\infty} = 0] - \delta_a - \delta_b.$$

Indeed, we have just proved that for the parameters fixed as above, $A \cap B \cap C$ implies D , for $m_0 = m_b$. Since we are taking the limit of a strictly increasing sequence, the assertion follows (and the limit exists, since the sequence is bounded).

Since the above inequality holds for all $\delta_a, \delta_b > 0$, it must also hold for $\delta_a = \delta_b = 0$. Thus, all that remains is to prove that

$$\lim_{m_0 \rightarrow \infty} \Pr[Z_m \leq 2^{-\ell^{\beta \cdot m}} \text{ for all } m \geq m_0] \leq \Pr[Z_\infty = 0] .$$

Assume to the contrary that there exists $0 < \beta < E$ and m_0 such that

$$\Pr[Z_m \leq 2^{-\ell^{\beta \cdot m}} \text{ for all } m \geq m_0] > \Pr[Z_\infty = 0] .$$

Clearly, this implies that

$$\Pr[\lim_{m \rightarrow \infty} Z_m = 0] > \Pr[Z_\infty = 0] .$$

Hence,

$$\Pr[\lim_{m \rightarrow \infty} Z_m = Z_\infty] < 1 ,$$

contradicting that fact that the sequence Z_m converges almost surely to Z_∞ .

III. PROOF OF LEMMA 1

We now explain why Lemma 2 implies Lemma 1. That is, why (3) implies (2). Clearly, (3) implies

$$\liminf_{m \rightarrow \infty} \Pr[Z_m \leq 2^{-\ell^{\beta \cdot m}}] \geq \Pr[Z_\infty = 0] .$$

Thus, the claim will follow if we prove that

$$\limsup_{m \rightarrow \infty} \Pr[Z_m \leq 2^{-\ell^{\beta \cdot m}}] \leq \Pr[Z_\infty = 0] .$$

Assume to the contrary that there exists $0 < \beta < E$ such that

$$\limsup_{m \rightarrow \infty} \Pr[Z_m \leq 2^{-\ell^{\beta \cdot m}}] > \Pr[Z_\infty = 0] .$$

The above implies that the Z_m cannot converge in probability to Z_∞ [6, page 70, Equation (5)]. This contradicts the fact that almost sure convergence implies convergence in probability [6, Theorem 4.2].

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